V. *On the Independence of the analytical and geometrical Methods of Investigation; and on the Advantages to be derived from their Separation. By* Robert Woodhouse, *A. M. Fellow of Caius College*, Cambridge. *Communicated by* Joseph Planta, *Esq. Sec. R. S.*

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One of the objects of the paper which last year I had the honour of presenting to the Royal Society, was to shew the insufficiency in mathematical reasoning, of a principle of analogy, by which the properties demonstrated for one figure were to be transferred to another, to which the former was supposed to bear a resemblance; and the argument for the insufficiency of the principle was this, that the analogy between the two figures was neither antecedent to calculation, nor independent of it, and consequently could not regulate it; that analogy was the object of investigation, not the guide; the result of demonstration, not its directing principle.

Having shewn that analogy could not establish the truth of certain mathematical conclusions, I next endeavoured to shew why such conclusions had been rightly inferred; not by proposing any new excogitated principle, nor by pointing out an hitherto unobserved intellectual process; but I conceived they might be obtained by operations conducted in a manner similar to that by which all reasoning with general terms is conducted,

and that the relations between the symbols or general terms were to be established by giving the true meaning to the connecting signs, which indicate not so much the arithmetical computation of quantities, as certain algebraical operations. It was further observed, that, from certain established formulas, abridged symbols or general terms might be formed, which consequently must have their signification dependent on such formulas; and that, although the parts of certain abridged expressions could not separately be arithmetically computed, yet the expressions themselves might be legitimately employed in all algebraic operations.

The chief object of my paper was to shew, that operations with imaginary quantities, as they are called, were strictly and logically conducted, that is, conducted after the same manner as operations with quantities that can be arithmetically computed: the question, whether calculation with imaginary symbols is commodious or not, was then slightly discussed. I have since attentively considered it, and, what usually happens in such cases, my inquiries have been extended beyond their original object; for, actual research has convinced me of what there were antecedent reasons for suspecting, that not only in the theory of angular functions, demonstration is most easy and direct by giving to quantities their true and natural* representation; but, that the introduction of expressions and formulas not analytical, into analytical investigation, has caused much ambiguity, confused notion, and paradox; that it has made

* $(2^{-1}).\left\{\varepsilon^{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}\right\}$, $(2\sqrt{-1})^{-1}.\left\{\varepsilon^{x\sqrt{-1}}-\varepsilon^{-x\sqrt{-1}}\right\}$ &c. I call the *natural* representations of the cosines, sines, &c. of an arc x; because, admitting the algebraical notation, they, by strict inference, adequately, unambiguously, and solely, represent the cosines, sines, &c.

demonstration prolix, by rendering it less direct, and has made it deficient in precision and exactness, by diverting the mind from the true source and derivation of analytical expression.

The expressions and formulas alluded to are geometrical, that is, taken from the language of geometry, and established by its rules: they are to be found mixed with analytical* expressions and reasonings, in all works on abstract science; and, as they are certainly foreign and circumlocutory, if it can be shewn that they are not essentially necessary, there will exist an argument for their exclusion, especially if it appears that in analytical investigation they are productive of the evils above mentioned.

That, in algebraical calculation, geometrical expressions and formulas are not essentially necessary, perhaps this short and easy consideration may convince us; since algebra is an universal language, it ought surely to be competent to express the conditions belonging to any subject of inquiry; and, if adequate expressions be obtained, then there is no doubt that with such, reasoning or deduction may be carried on.

All expressions and formulas, such as, sin. x, cos. x, hyp. log. x, sin. $nx = 2$ cos. x. sin. $(n-1)\ x -$ sin. $(n-2)\ x$,

* The terms analysis, analytical, algebra, algebraical, have been so often distinguished, and so often confounded, that I shall not take the trouble again to distinguish them. I use the words analytical, algebraical, indifferently, in contradistinction to geometrical. The first relates to an arbitrary system of characters; the latter to a system of signs, that are supposed to bear a resemblance to the things signified, and in which system, lines and diagrams are used as the representatives of quantity: and I am principally induced to use the words indifferently, because, if analytical were properly defined, another word with a sufficient extent of meaning could not be found; for, by an improper limitation, the word algebraical has not an extensive signification, being frequently used in contradistinction to transcendental, exponential, &c.

$\int \dot{x} \cdot (1 - x^2)^{\frac{1}{2}} =$ circular arc, $\int \dot{x} \cdot \sqrt{\frac{1 + n^2 x^2}{1 - x^2}} =$ elliptical arc, &c. are geometrical, or involve geometrical language: they suppose the existence of a particular system of signs, and method of deduction; and relate to certain theorems, established conformably to such system and method.

I. Sin. x, cos. x, tang. x, &c. These expressions are borrowed from geometry; but, analytically, denote certain functions of x. Typographically considered, these expressions are more commodious than $(2\sqrt{-1})^{-1}\left\{\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}}\right\}$, $(2)^{-1}\left\{\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}}\right\}$, &c. but this is the sole advantage; for, all analytical operations with these latter signs are much easier, and more expeditious, than with the former; since they are carried on after a manner analogous to that by which operations with similar expressions are. But, if the geometrical expressions be retained, then, in order to calculate with them, recourse must be had to the geometrical method, proceeding by the similarity of triangles, the doctrines of proportions, and of prime and ultimate ratios; so that, in the same investigation, two methods of deduction, between which there is no similarity, must be employed.

II. The value of $\int \dot{x} \cdot (1 + x)^{-1}$, is said to be a portion of the area of an hyperbola intercepted between two ordinates to its assymptotes; but this is a foreign and circumlocutory mode of expression; since, to find the value of the area, $\dot{x} \cdot (1 + x)^{-1}$ must be expanded, and the integrals of the several terms taken; and this same operation must have taken place, in order to approximate to the value of $\int \dot{x} \cdot (1 + x)^{-1}$, if no such curve as the hyperbola had ever been invented.

III. $\int \dot{x} \cdot \left\{1 - x^2\right\}^{-\frac{1}{2}}$ is said to equal the arc of the circle

rads. 1, sin. x; but nothing is gained by this; since, in order to find the arc of a circle, $\dot{x}\cdot(1-x^2)^{-\frac{1}{2}}$ is expanded, and the integrals of the several parts taken and added together. To shew (if it is necessary to add any thing more on so clear a point) that $\int\dot{x}\cdot\left\{1-x^2\right\}^{-\frac{1}{2}}=$ arc circle, is merely a mode of expression borrowed from geometry; suppose the investigation of the properties of motion to have been prior to the investigation of the properties of extension, for, that the science of geometry was first invented is properly an accidental circumstance, then, such an expression as $\int\dot{x}\cdot\left\{1-x^2\right\}^{-\frac{1}{2}}$ might have occurred, and its value must have been exhibited as it *really* is now, that is, by expanding it, and integrating the several terms.

IV. It is an objection certainly against these modes of expression, that they are foreign, and tend to produce confused and erroneous notions; for the student may be led by them to believe, that the determination of the values of certain analytical expressions, essentially require the existence of certain curves, and the investigation of their properties. But there is a more valid objection against them, which is, that they divert the mind from the true derivation of such expressions as $\frac{\dot{x}}{x}$, $\dot{x}\cdot(1-x^2)^{-\frac{1}{2}}$, &c. and consequently tend to produce ambiguity and indirect methods; for although, in order to obtain approximately the numerical value of $\int\frac{\dot{x}}{x}$, $\int\dot{x}\cdot(1-x^2)^{-\frac{1}{2}}$, &c. it is convenient to expand the expressions, and to take the integrals of the resulting terms, yet, if the symbol $\int$ denotes a reverse operation, $\int\frac{\dot{x}}{x}$, $\int\dot{x}\cdot(1-x^2)^{-\frac{1}{2}}$ are not properly and by strict inference equal to $(x-1)-\frac{1}{2}\left\{x-1\right)^2+\frac{1}{3}\left\{x-1\right)^3-$, &c. and $x+\frac{x^3}{2.3}+\frac{3x^5}{5.8}+$, &c. But, in order to explain clearly what I mean, it is

necessary to state what I understand by the integral or fluent of an expression.

V. Let φx denote a function of x; if x be increased by o, then φx becomes $\varphi(x+o)$, and $\varphi(x+o)$, developed according to the powers of o, becomes $\varphi x + Po + \frac{Q}{1.2}o^2 + \frac{R}{1.2.3}o^3$, &c. where P is derived from φx, Q from P, R from Q, &c. by the same law; so that the manner of deriving P being known, Q, R, &c. are known. The entire difference or increment of φx is $\varphi(x+o) - \varphi x$; the differential or fluxion of φx is only a part of the difference or $P.o$. If, instead of o, dx, or $\dot{x}$, be put, it is $P.dx$ or $P\dot{x}$; the integral or fluent of $P\dot{x}$ is that function from which $P\dot{x}$ is derived; and, in order to remount to it, we must observe the manner or the operation by which it was deduced; and, by reversing such operation, the integral or fluent is obtained. Now, in taking the fluxion of certain functions of x, it appears there are conditions to which the indices of x without and under the vinculum are subject: hence, whether or not a proposed fluxion can have its fluent assigned, we must see if the fluxion has the necessary conditions. Expressions such as $\frac{\dot{x}}{x}$, $\frac{\dot{x}}{1+x}$, $\frac{\dot{x}}{\sqrt{1-x^2}}$, &c. have not these conditions; and consequently there is no function φx of x, such that the second term of the developement of $\varphi(x+\dot{x})$ is equal either to $\frac{\dot{x}}{x}$, or $\frac{\dot{x}}{1+x}$, or $\frac{\dot{x}}{\sqrt{1-x^2}}$, or, &c. There are, however, integral equations from which such expressions may be derived. Thus, let $x = \varepsilon^z$, then $\frac{\dot{x}}{x} = \dot{z}$, let $1 + x = \varepsilon^z$ $\therefore$ $\frac{\dot{x}}{1+x} = \dot{z}$, let $x = \frac{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}}{2\sqrt{-1}}$ $\therefore$ $\frac{\dot{x}}{\sqrt{1-x^2}} = \dot{z}$

Now, from these equations, the differential equations $\frac{\dot{x}}{x} = \dot{z}$, $\frac{\dot{x}}{1+x} = \dot{z}$, $\frac{\dot{x}}{\sqrt{1-x^2}}$, &c. may, by expunging the exponential

quantities, be derived; consequently, if the symbol $\int$ is to designate a reverse operation, I can only know what that reverse operation is, by attending to the manner by which the expressions affected with the symbol $\int$ were derived. Hence,

VI. $\int \frac{\dot{x}}{x} = z$ when $x = \varepsilon^{z}$.

$\int \frac{\dot{x}}{1+x} = z$ when $1 + x = \varepsilon^{z}$.

$\int \frac{\dot{x}}{\sqrt{1-x^2}} = z$ when $x = (2\sqrt{-1})^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\}$.

In like manner,

$\int \dot{x}\cdot(1+x^2)^{-\frac{1}{2}} = z$, $x + \sqrt{1+x^2} = \varepsilon^{z}$ or $x = \frac{\varepsilon^{z} - \varepsilon^{-z}}{2}$.*

$\int \dot{x}\cdot(2x+x^2)^{-\frac{1}{2}} = z$, $1 + x + \sqrt{2x+x^2} = \varepsilon^{z}$.

$\int \frac{2\dot{x}}{1-x^2} = z$, $\frac{1+x}{1-x} = \varepsilon^{z}$ or $x = \frac{\varepsilon^{z}-1}{\varepsilon^{z}+1}$.

$\int \frac{2\dot{x}}{x\sqrt{1+x^2}} = z$, $\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} = \varepsilon^{z}$ or $\sqrt{1+x^2} = \frac{1+\varepsilon^{z}}{1-\varepsilon^{z}}$,

or $x = \frac{2}{\varepsilon^{-\frac{z}{2}} - \varepsilon^{\frac{z}{2}}}$..

Again, suppose

$x = \left\{2\sqrt{-1}\right\}^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\} \therefore \dot{x} = 2^{-1}\dot{z}\cdot\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{z\sqrt{-1}}\right\}$,

but $\sqrt{1-x^2} = 2^{-1}$, $(\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\}$; consequently $\dot{x} = \dot{z}\sqrt{1-x^2}$,

or $\dot{z} = \frac{\dot{x}}{\sqrt{1-x^2}}$: hence, reversely,

$\int \frac{\dot{x}}{\sqrt{1-x^2}} = z$, x being $= (2\sqrt{-1})^{-1}.\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\}$.

In like manner,

$\int \frac{-\dot{x}}{\sqrt{1-x^2}} = z$, $x = 2^{-1}.\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}$.

$\int \frac{\dot{x}}{\sqrt{2x-x^2}} = z$, $x = \left(1 - 2^{-1}.\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}\right)$.

* I take no notice, at present, of the arbitrary quantities which may be introduced in the integration of these equations.

$$\int \frac{x^{\cdot}}{1-x^2} = z,\ x = \frac{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}}{\sqrt{-1}\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}},\ \text{or}\ \frac{\varepsilon^{2z\sqrt{-1}} - 1}{\sqrt{-1}\,(\varepsilon^{2z\sqrt{-1}} + 1)}.$$

$$\int \frac{x^{\cdot}}{x\sqrt{x^{\cdot 2}-1}} = z,\quad x = \frac{2}{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}}.$$

And a variety of forms may be obtained, by substituting for x different functions of x, in the expression $\frac{x^{\cdot}}{\sqrt{1-x^2}}$. Hence, if the symbol $\int$ is made to denote a reverse operation, the integral equations of the preceding differential equations have been rightly assigned. All other methods of assigning the integrals, by the properties of logarithms, by circular arcs, by logarithmic and hyperbolic curves,* are indirect, foreign, and ambiguous.

VII. An instance or two will shew the advantage of adhering to the true and natural derivation of analytical expressions. Let x and y be the co-ordinates of a circle; then, $1 = x^2 + y^2$, and $y = \sqrt{(1-x^2)}$, now (arc)$^{\cdot}$ or $z^{\cdot} = \sqrt{(x^{\cdot 2} + y^{\cdot 2})}$ $=$, in this instance, $x^{\cdot}\,(1-x^2)^{-\frac{1}{2}}$: but it has appeared, that if $x = \left\{2\sqrt{-1}\right\}^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\}$, $z^{\cdot} = x^{\cdot}\,(1-x^2)^{-\frac{1}{2}}$; consequently, in a circle, the co-ordinate x, or, in the language of trigonometry, the sine $x =$ developement of

$$(2\sqrt{-1})^{-1}.\ \left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\} = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \text{\&c.}$$

and y or cosine $= 2^{-1}.\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\} = 1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \text{\&c.}$

1. This method of determining the series for the sine in terms of the arc, is, I think, simple, direct, and exact; it requires no assumption of a series with indeterminate coefficients, nor

* By the strange way of determining the meaning and value of analytical expressions from geometrical considerations, it should seem, as if certain curves were believed to have an existence independent of arbitrary appointment.

any preparatory process to shew that the value of the first coefficient must $= 1$.*

VIII. EULER demonstrated this formula to be true, viz. $\frac{\text{Arc}}{2} = \sin.\ \text{arc} - \frac{1}{2}\sin.\ 2\,\text{arc} + \frac{1}{3}\sin.\ 3\,\text{arc} - \frac{1}{4}\sin.\ 4\,\text{arc} + \&c.$ The following is its analytical deduction,

$$z \cdot = z \cdot \left\{ \frac{\varepsilon^{z\sqrt{-1}} + 1}{\varepsilon^{z\sqrt{-1}} + 1} \right\} = z \cdot \left\{ \frac{\varepsilon^{z\sqrt{-1}}}{\varepsilon^{z\sqrt{-1}} + 1} \right\} + z \cdot \left\{ \frac{1}{\varepsilon^{z\sqrt{-1}} + 1} \right\}$$

$$= z \cdot \left\{ \frac{\varepsilon^{z\sqrt{-1}}}{\varepsilon^{z\sqrt{-1}} + 1} \right\} + z \cdot \left\{ \frac{\varepsilon^{z-\sqrt{-1}}}{\varepsilon^{-z\sqrt{-1}} + 1} \right\}$$

$$= z \cdot \left\{ \begin{array}{l} \varepsilon^{z\sqrt{-1}} - \varepsilon^{2z\sqrt{-1}} + \varepsilon^{3z\sqrt{-1}} - \&c. \\ + \varepsilon^{-z\sqrt{-1}} - \varepsilon^{-2z\sqrt{-1}} + \varepsilon^{-3z\sqrt{-1}} - \&c. \end{array} \right\}$$

$$\therefore z = \frac{1}{\sqrt{-1}} \left\{ \begin{array}{l} \varepsilon^{z\sqrt{-1}} - \frac{\varepsilon^{2z\sqrt{-1}}}{2} + \frac{\varepsilon^{3z\sqrt{-1}}}{3} - \&c. \\ - \varepsilon^{-z\sqrt{-1}} + \frac{\varepsilon^{-2z\sqrt{-1}}}{2} - \frac{\varepsilon^{-3z\sqrt{-1}}}{3} + \&c. \end{array} \right\}$$

and $\frac{z}{2} = (2\sqrt{-1})^{-1} \cdot \left\{ \varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}} \right\} - \frac{1}{2}(2\sqrt{-1})^{-1}.$ $\left\{ \varepsilon^{2z\sqrt{-1}} - \varepsilon^{-2z\sqrt{-1}} \right\}. + \frac{1}{3}(2\sqrt{-1})^{-1}\left\{ \varepsilon^{3z\sqrt{-1}} - \varepsilon^{3z\sqrt{-1}} \right\} - \&c.$ which is the analytical translation of EULER's formula.

IX. EULER likewise shewed that

$$\sin.\ x = 2^n.\ \cos.\frac{x}{2}.\ \cos.\frac{x}{4}.\ \cos.\frac{x}{8} \ldots\ldots \cos.\frac{x}{2^n} \cdot \sin.\frac{x}{2^n}.$$

Which may be thus demonstrated,

$$\sin.\ x = (2\sqrt{-1})^{-1}\left\{ \varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}} \right\};$$

but $(2\sqrt{-1})^{-1}\left\{ \varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}} \right\} = 2 \,.\, 2^{-1}\left\{ \varepsilon^{\frac{x}{2}\sqrt{-1}} + \varepsilon^{-\frac{x}{2}\sqrt{-1}} \right\},$

$$(2\sqrt{-1}\}^{-1}\ \varepsilon^{\frac{x}{2}\sqrt{-1}} - \varepsilon^{-\frac{x}{2}\sqrt{-1}}\}$$

$$= 2 \,.\, 2^{-1}\left(\varepsilon^{\frac{x}{2}\sqrt{-1}} + \varepsilon^{-\frac{x}{2}\sqrt{-1}}\right\} \,.\, 2 \,.\, 2^{-1}\left(\varepsilon^{\frac{x}{4}\sqrt{-1}} + \varepsilon^{-\frac{x}{4}\sqrt{-1}}\right).$$

$$(2\sqrt{-1})^{-1}.\left\{ \varepsilon^{\frac{x}{4}\sqrt{-1}} - \varepsilon^{-\frac{x}{4}\sqrt{-1}} \right\}$$

* See LAGRANGE, Fonctions Analytiques. p. 26. LACROIX, Traité du Calcul. differentiel, &c. p. 56. LE SEUR, Sur le Calcul. diff. p. 105. EULER, Anal. Inf. Art. 133, 134.

$$= 2 \, . \, 2^{-1}\left(\varepsilon^{\frac{x}{2}\sqrt{-1}} + \varepsilon^{-\frac{x}{2}\sqrt{-1}}\right) . \, 2 \, . \, 2^{-1}\left(\varepsilon^{\frac{x}{4}\sqrt{-1}} + \varepsilon^{-\frac{x}{4}\sqrt{-1}}\right)$$

$$. \, 2 \, . \, 2^{-1}\left\{\varepsilon^{\frac{x}{8}\sqrt{-1}} + \varepsilon^{-\frac{x}{8}\sqrt{-1}}\right\} \left\{2\sqrt{-1}\right\}^{-1} \left(\varepsilon^{x\sqrt{-1}} - \varepsilon^{-\frac{x}{8}\sqrt{-1}}\right),$$

or, generally,

$$= 2^n . \, 2^{-1}\left\{\varepsilon^{\frac{x}{2}\sqrt{-1}} + \varepsilon^{-\frac{x}{2}\sqrt{-1}}\right\} . \, 2^{-1}\left\{\varepsilon^{\frac{x}{4}\sqrt{-1}} + \varepsilon^{-\frac{x}{4}\sqrt{-1}}\right\} . \, 2^{-1}$$

$$\left(\varepsilon^{\frac{x}{8}\sqrt{-1}} + \varepsilon^{-\frac{x}{8}\sqrt{-1}}\right\} \ldots .$$

$$. \, . \, 2^{-1}\left\{\varepsilon^{\frac{x}{2^n}\sqrt{-1}} + \varepsilon^{-\frac{x}{2^n}\sqrt{-1}}\right\} . \left(2\sqrt{-1}\right)^{-1}\left\{\varepsilon^{\frac{x}{2^n}\sqrt{-1}} - \varepsilon^{-\frac{x}{2^n}\sqrt{-1}}\right).$$

Which is the analytical translation of sin. $x = 2^n$. cos. $\frac{x}{2}$. cos. $\frac{x}{4}$ &c. Euler, and after him other authors, have demonstrated these formulas by the aid of logarithms, and of theorems drawn from geometry.

X. Euler and Lagrange have treated certain differential equations, which are said to admit for their complete integration an algebraic form, although the integration of each particular term depends on the quadrature of the circle and hyperbola. I purpose to integrate these differential equations, by the method adopted in Articles V. VI.

Let fx, fy, denote functions of x and y.

Suppose the differential equation to be

$\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = o$; then $fx + fy = a$ when $x = \varepsilon^{fx}$, $y = \varepsilon^{fy}$. Hence, $xy = \varepsilon^{fx+fy} = \varepsilon^{a} = \mathrm{A}$, a constant quantity.

2dly. Let $\frac{\dot{x}}{\sqrt{1-x^2}} + \frac{\dot{y}}{\sqrt{1-y^2}} = o$

$\therefore fx + fy = a$, x being $= \left\{2\sqrt{-1}\right\}^{-1} . \left(\varepsilon^{fx\sqrt{-1}} - \varepsilon^{-fx\sqrt{-1}}\right)$,

and $y = 2\sqrt{-1})^{-1} . \left(\varepsilon^{fy\sqrt{-1}} - \varepsilon^{-fy\sqrt{-1}}\right)$; or $\sqrt{(1-x^2)} = 2^{-1}$. $\left(\varepsilon^{fx\sqrt{-1}} + \varepsilon^{-fx\sqrt{-1}}\right)$, and $\sqrt{1-y^2} = 2^{-1} . \left(\varepsilon^{fy\sqrt{-1}} + \varepsilon^{-fy\sqrt{-1}}\right)$.

Hence, $x . \sqrt{(1-y^2)} + y\sqrt{(1-x^2)}$

$$= \left(2\sqrt{-1}\right)^{-1} . \left\{\varepsilon^{(fx+fy)\sqrt{-1}} - \varepsilon^{-(fx+fy)\sqrt{-1}}\right\}$$

$$= \left(2\sqrt{-1}\right)^{-1} . \left\{\varepsilon^{a\sqrt{-1}} - \varepsilon^{-a\sqrt{-1}}\right\} = \mathrm{A}, \text{ a constant quantity.}$$

3dly. Let $\frac{x^{\cdot}}{\sqrt{a+bx+cx^2}} + \frac{y^{\cdot}}{\sqrt{a+by+cy^2}} = 0$

$\therefore \frac{x^{\cdot}}{\sqrt{c}\sqrt{x^2+\frac{bx}{c}+\frac{a}{c}}} + \frac{y^{\cdot}}{\sqrt{c}\sqrt{\left(y^2+\frac{by}{c}+\frac{a}{c}\right)}} = 0.$

Let $x + \frac{b}{2c} = v$, $y + \frac{b}{2c} = v'$ and $r^2 = \frac{a}{c} - \frac{b^2}{4c^2}$

$\therefore \frac{v^{\cdot}}{\sqrt{c}\sqrt{v^2+r^2}} + \frac{v'^{\cdot}}{\sqrt{c}\sqrt{v'^2+r^2}} = 0,$

taking the integrals

$c^{-\frac{1}{2}}\left\{V + V'\right\} = \alpha$, $v = \frac{\varepsilon^{V} - r^2\varepsilon^{-V}}{2}$, $v' = \frac{\varepsilon^{V'} - r^2\varepsilon^{-V'}}{2}$

$\therefore v\sqrt{r^2+v'^2} + v'\sqrt{(r^2+v^2} = \frac{\varepsilon^{V+V'} - r^4\varepsilon^{-(V+V')}}{2} = \frac{\varepsilon^{\alpha\sqrt{c}} - r^4\varepsilon^{-\alpha\sqrt{c}}}{2}$

$=$ A, and restoring the values of x and y,

$\frac{2cx+b}{c}\sqrt{(a+by+cy^2)} + \frac{2cy+b}{c}\sqrt{(a+bx+cx^2)} = A'.$

By the above operation it appears, that certain algebraical expressions, as $x\sqrt{1-y^2}+y\sqrt{1-x^2}$, $\frac{2cx+b}{c}\sqrt{a+by+cy^2}$ &c. may be deduced, which answer the equations $\int\frac{x^{\cdot}}{\sqrt{1-x^2}} + \int\frac{y^{\cdot}}{\sqrt{1-y^2}}$ &c. But, strictly speaking, such algebraical expressions are not the integrals: they are rather expressions deduced from the true integral equations, from which other algebraical expressions, besides those put down, might be deduced.*

* For the integration of this sort of differential equations, see Mem. de Turin. Vol. IV. p. 98. "Sur l'Integration de quelques Equations differentielles, dont les indeterminées sont separées, mais dont chaque Membre en particulier n'est point integrable." In this Memoir are given three different methods of integrating $x^{\cdot}(1-x^2)^{-\frac{1}{2}} = y^{\cdot}(1-y^2)^{-\frac{1}{2}}$; by circular arcs and certain trigonometrical theorems, by impossible logarithms, and by partial integrations. Strictly speaking, all these methods are indirect; and the two first are only different but circuitous modes of expressing the method given in Art. X. See likewise Euler, Calc. integral Vol. II. Novi Comm. Petrop. Tom. VI. p. 37. Tom. VII. p. 1. It is to be observed, that in the present state of analytic science, there is no certain and direct method of integrating differential equa-

XI. In the irreducible case of cubic equations, the root, it is said, may be exhibited by means of certain lines drawn in a circle. There is, however, independently of all geometrical considerations, a method of analytically expressing the root; and, from the analytical expression, although it is not the formula which from the time of CARDAN mathematicians have been seeking, the value of the root may in all cases be arithmetically computed; but, previously, it is necessary to shew what are the different symbols that may be substituted for z in the equations, $x = (2\sqrt{-1})^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\}$, and $\sqrt{(1-x^2)} = 2^{-1}$ $\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}$. Let $x = 1$, and π be the value of z that answers the equations $1 = (2\sqrt{-1})^{-1}\left\{\varepsilon^{\pi\sqrt{-1}} - \varepsilon^{-\pi\sqrt{-1}}\right\}$ and $0 = 2^{-1}(\varepsilon^{\pi\sqrt{-1}} + \varepsilon^{-\pi\sqrt{-1}})$, which value of π may be numerically computed from the expression .. $\pi = z = x + \frac{x^3}{2.3} + \frac{3x^5}{5.8} + \frac{5x^7}{7.16} +$ &c. $(x = 1)$.

Hence, $\varepsilon^{\pi\sqrt{-1}} = -\varepsilon^{-\pi\sqrt{-1}} = \sqrt{-1}$ $\therefore \varepsilon^{2\pi\sqrt{-1}} = \varepsilon^{-2\pi\sqrt{-1}} = -1$ $\therefore \varepsilon^{4\pi\sqrt{-1}} = \varepsilon^{-4\pi\sqrt{-1}} = 1$ $\therefore$, $\varepsilon^{8\pi\sqrt{-1}} = \varepsilon^{-8\pi\sqrt{-1}} = 1$ $\therefore$ $\varepsilon^{16\pi\sqrt{-1}} = \varepsilon^{-16\pi\sqrt{-1}} = 1$ &c. (for since $\varepsilon^{-m\pi\sqrt{-1}} = \frac{1}{\varepsilon^{m\pi\sqrt{-1}}}$, and $\varepsilon^{m\pi\sqrt{-1}} = \varepsilon^{-m\pi\sqrt{-1}} = \frac{1}{\varepsilon^{m\pi\sqrt{-1}}}$ $\therefore \varepsilon^{2m\pi\sqrt{-1}} = 1$).

Again, since $\varepsilon^{4\pi\sqrt{-1}} = 1$ and $\varepsilon^{8\pi\sqrt{-1}} = 1$, $\varepsilon^{12\pi\sqrt{-1}} = 1$; and

tions such as $x\cdot\left\{a+bx+cx^2+dx^3+\epsilon x^4\right\}^{-\frac{1}{2}} + y\cdot\left\{a+by+cy^2+dy^3+\epsilon y^4\right\}^{-\frac{1}{2}} = 0$, because no analytical expression or equation of a finite form has hitherto been invented, from which, according to the processes of the differential Calculus, such differential equations may be deduced. To find the algebraical expressions which answer to these equations, recourse must be had to what are properly to be denominated artifices. For such, see Mem. de Turin. Vol. IV. Comm. Petr. Tom. VI. VII. LAGRANGE, Fonct. Analyt. p. 80. LACROIX, Calc. diff. p. 427, &c.

generally $\varepsilon^{4n\pi\sqrt{-1}} = \varepsilon^{-4n\pi\sqrt{-1}} = 1$, n any number of the progression 0, 1, 2, 3, 4, &c.

And, since $\varepsilon^{2\pi\sqrt{-1}} = \varepsilon^{-2\pi\sqrt{-1}} = -1 \therefore \varepsilon^{2\pi\sqrt{-1}} \times \varepsilon^{4n\pi\sqrt{-1}} = \varepsilon^{-2\pi\sqrt{-1}} \times \varepsilon^{-4n\pi\sqrt{-1}} = -1$, or $\varepsilon^{(2n+1)\,2\pi\sqrt{-1}} = \varepsilon^{-(2n+1)2\pi\sqrt{-1}} = -1$, n any number of the progression 0, 1, 2, 3, 4, 5, &c.

Hence it appears, that if $x = (2\sqrt{-1})^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\}$, $x \times 1 = \left\{2\sqrt{-1}\right\}^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\} \times \varepsilon^{4n\pi\sqrt{-1}} =$ (since $\varepsilon^{4n\pi\sqrt{-1}} = \varepsilon^{-4n\pi\sqrt{-1}}$) $(2\sqrt{-1})^{-1}\left\{\varepsilon^{(4n\pi+z)\sqrt{-1}} - \varepsilon^{-(4n\pi+z)\sqrt{-1}}\right\}$.

Again, since $\varepsilon^{(2n+1)2\pi\sqrt{-1}} = \varepsilon^{-(2n+1)\,2\pi\sqrt{-1}} = -1$

$x \times -1 = (2\sqrt{-1})^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\}\varepsilon^{(2n+1)\pi\sqrt{-1}}$

$= (2\sqrt{-1})^{-1}\left\{-\varepsilon^{((2n+1)2\pi-z)\sqrt{-1}} - \varepsilon^{-((2n+1)2\pi-z)\sqrt{-1}}\right.$,

consequently,

$x = (2\sqrt{-1})^{-1}\left\{\varepsilon^{((2n+1)2\pi-z)\sqrt{-1}} - \varepsilon^{-((2n+1)2\pi-z)\sqrt{-1}}\right\}$,

or the equation $x = (2\sqrt{-1})^{-1}\left\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\right\}$ is true, when instead of z is put $(4\pi+z)$ or $(8\pi+z)$, or generally $(4n\pi+z)$; and is moreover true, when instead of z is put

$(2\pi-z)$, $(6\pi-z)$, or generally $(2n+1)\ 2\pi-z$.

In like manner, the equation $\sqrt{1-x^2} = 2^{-1}\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}$ is true, when instead of z is put

$4\pi+z$, $8\pi+z$, or $12\pi+z$, or generally $4n\pi+z$;

and is moreover true, when instead of z is put

$4\pi-z$, ,$8\pi-z$, or $12\pi-z$, or generally $4n\pi-z$.

Let now $x^3-qx=r$, then, by CARDAN'S solution,

$$x = \sqrt[3]{\left(\frac{r}{2} + \sqrt{\left(\frac{r}{4} - \frac{q^3}{27}\right)}\right)} + \sqrt[3]{\left(\frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^4}{27}\right)}\right)};$$

put $\frac{r}{2} = a$, $\frac{r^2}{4} - \frac{q^3}{27} = -b^2$, then $x = \sqrt[3]{(a+b\sqrt{-1})} + \sqrt[3]{a(-b\sqrt{-1})}$.

Let $a + b\sqrt{-1} = m\varepsilon^{z\sqrt{-1}} \therefore a - b\sqrt{-1} = m\varepsilon^{-z\sqrt{-1}}$

$a^2+b^2=m^2$, $a=m\left\{\frac{\varepsilon^{z\sqrt{-1}}+\varepsilon^{-z\sqrt{-1}}}{2}\right\}$, $b=m\left\{\frac{\varepsilon^{z\sqrt{-1}}-\varepsilon^{-z\sqrt{-1}}}{2\sqrt{-1}}\right\}$, or $2^{-1}.\left\{\varepsilon^{z\sqrt{-1}}+\varepsilon^{-z\sqrt{-1}}\right\}=\frac{a}{\sqrt{a^2+b^2}}$, and $(2\sqrt{-1})^{-1}\left\{\varepsilon^{z\sqrt{-1}}-\varepsilon^{-z\sqrt{-1}}\right\}=\frac{b}{\sqrt{a^2+b^2}}$; but, from what has been premised, these equations are true, when instead of z is put $\theta+z$, or $2\theta+z$, or $4\theta+z$, or generally $n\theta+z$, $(4\pi=\theta)$.

Hence, $\sqrt[3]{(a+b\sqrt{-1})}+\sqrt[3]{(a-b\sqrt{-1})}$ is $m^{\frac{1}{3}}\left\{\varepsilon^{\frac{z}{3}\sqrt{-1}}+\varepsilon^{-\frac{z}{3}\sqrt{-1}}\right\}$, or $m^{\frac{1}{3}}\left\{\varepsilon^{\frac{\theta+z\sqrt{-1}}{3}}+\varepsilon^{\frac{-\theta+z\sqrt{-1}}{3}}\right\}$, or generally $m^{\frac{1}{3}}\left\{\varepsilon^{\frac{n\theta+z\sqrt{-1}}{3}}+\varepsilon^{\frac{-(n\theta+z)\sqrt{-1}}{3}}\right\}$: there are, however, only 3 different values of x, for the index of ε in the fourth value is $\frac{3\theta+z\sqrt{-1}}{3}$, and $\varepsilon^{\frac{3\theta+z\sqrt{-1}}{3}}=\varepsilon^{\theta\sqrt{-1}}\times\varepsilon^{\frac{z}{3}\sqrt{-1}}=1\times\varepsilon^{\frac{z}{3}\sqrt{-1}}$ $\therefore$ the fourth value is the same as the first. Again, the index of ε in the fifth value is $\frac{4\theta+z}{3}\sqrt{-1}$; but $\varepsilon^{\frac{(4\theta+z)}{3}\sqrt{-1}}=\varepsilon^{\theta\sqrt{-1}}\times\varepsilon^{\frac{\theta+z}{3}\sqrt{-1}}=1\times\varepsilon^{\frac{\theta+z}{3}\sqrt{-1}}$. The 5th value is the same as 2d, and so on; and, consequently, the indices of e in the 3 different values of x are $\pm\frac{z}{3}\sqrt{-1}$, $\pm\frac{\theta+z}{3}\sqrt{-1}$, $\pm\frac{2\theta+z}{3}\sqrt{-1}$.

If, instead of the index of ε in the 3d value, $\pm\frac{\theta-z}{3}\sqrt{-1}$ be put, the value of the root remains the same; for, since $\varepsilon^{\theta\sqrt{-1}}=\varepsilon^{-\theta\sqrt{-1}}=1$ $\therefore x\times 1=m^{\frac{1}{3}}\times\left\{\varepsilon^{\frac{2\theta+z}{3}\sqrt{-1}}\times\varepsilon^{-\theta\sqrt{-1}}+\varepsilon^{\frac{-2\theta+z}{3}\sqrt{-1}}\times\varepsilon^{\theta\sqrt{-1}}\right\}=m^{\frac{1}{3}}\left\{\varepsilon^{\frac{-\theta-z}{3}\sqrt{-1}}+\varepsilon^{\frac{\theta-z}{3}\sqrt{-1}}\right\}$.

This mode of representing the roots is not, as has been

already stated, according to the conditions* of the formula demanded by mathematicians. It enables us, however, immediately to ascertain that the roots are possible, and to calculate their approximate value; for, when $\sqrt{1-x^2}$ or $y = 2^{-1}$ $\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}$, $z = \int - \frac{y \cdot}{\sqrt{1-y^2}}$

$= \alpha - \left\{ y + \frac{y^3}{3.2} + \frac{3y^5}{5.8} + \frac{5y^7}{7.16} + \&c. \right\}$,

when $z = o$ $y = 2^{-1}\left\{\varepsilon^o + \varepsilon^{-o}\right\} = 1$

$\therefore \alpha = 1 + \frac{1}{3.2} + \frac{3}{5.8} + \frac{5}{7.16} + \&c. \} = \pi$.

Hence, we may numerically approximate to the value of z from the expression $z = \pi - \left\{ y + \frac{y^3}{3.2} + \frac{3y^5}{5.8} + \&c. \right\}$ when y is given, and < 1. Now, in the case of the cubic equation, $y = \frac{a}{\sqrt{a^2+b^2}} = \frac{3r\sqrt{3}}{2q\sqrt{q}}$; and, since $\frac{r^2}{4} < \frac{q^3}{27} \therefore \frac{3r\sqrt{3}}{2q\sqrt{q}}$ is < 1, consequently the value of z may be obtained; suppose it t, then the roots are to be approximated to, by means of the series that result from the developements of the forms by which they are represented; to wit,

$$2\sqrt{\frac{q}{3}}\left\{1 - \frac{t^2}{1.2.3^2} + \frac{1}{1.2.3.4} \cdot \frac{t^4}{3^4} - \&c. \right\}$$
$$2\sqrt{\frac{q}{3}}\left\{1 - \frac{(\theta+t)^2}{1.2.3^2} + \frac{1}{1.2.3.4} \cdot \frac{(\theta+t)^4}{3^4} - \&c. \right\}$$
$$2\sqrt{\frac{q}{3}}\left\{1 - \frac{(2\theta+t)^2}{1.2.3^2} + \frac{1}{1.2.3.4} \cdot \frac{(2\theta+t)^4}{3^4} - \&c. \right\}$$

Now these series converge; for, since t is finite, we must at length arrive at a term $\underset{n}{\mathrm{A}}$, in which $(n-1)\,n$ is $> \left(\frac{t}{3}\right)^2$; and, since $(n+1)$th term $\left(\underset{n+1}{\mathrm{A}}\right)$ is $= \underset{n}{\mathrm{A}} \times \left(\frac{t}{3}\right)^2 \times \frac{1}{n+1\,(n+2)} \therefore \underset{n+1}{\mathrm{A}}$ is

* The conditions of the formula are, that it should be finite in regard to the number of terms, free from imaginary quantities, and containing only the coefficients q and r. See Mem. de l'Acad. 1738.

$< \frac{A}{n}$, a fortiori, $\frac{A}{n+1}$ is $< \frac{A}{n+1}$, and so on; the terms after the $n-1$th term constantly diminishing.*

The above method is purely analytical: it has no tacit reference to other methods; it does not virtually suppose the existence either of an hyperbola or circle. If practical commodiousness, however, be aimed at, it is *convenient* to give a different expression to the values of the roots, or to translate them into geometrical language: and this, because tables have been calculated, exhibiting the numerical values of the cosines, &c. of circular arcs. Now, since it has already appeared that the cosine of an arc $z = 2^{-1}\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}$, the 3 roots of the equation $x^3 - qx = r$ may be said to equal

$$2\sqrt{\frac{q}{3}} \cdot \cos. \frac{t}{3},\ 2\sqrt{\frac{q}{3}} \cos. \frac{\theta+t}{3},\ 2\sqrt{\frac{q}{3}},\ \cos. \frac{\theta-t}{3}.$$

XII. In the fifth volume of his *Opuscules*,† D'ALEMBERT

* In the Phil. Trans. for 1801. p. 116, I mentioned M. NICOLE as the first mathematician who shewed the expression of the root in the irreducible case, when expanded, to be real. But the subjoined passage, in LEIBNITZ's Letter to WALLIS, causes me to retract my assertion. "Diu est quod ipse quoque judicavi $\sqrt{^3 a+b\sqrt{-1}}$ "$+\sqrt{^3 a+b\sqrt{1}} = x$ esse quantitatem realem, etsi speciem habeat imaginariæ; "ob virtualem nimirum imaginariæ destructionem, perinde ac in destructione actuali "$a+b\sqrt{-1} + a-b\sqrt{-1} = 2a$. Hinc, si ex $\sqrt{^3 a \pm b\sqrt{-1}}$ extrahamus radicem "ope seriei infinitæ, ad inveniendum valorem ipsius z serie tali expressum, efficere "possumus, ut reapse evanescat imaginaria quantitas. Atque ita etiam, in casu ima-"ginario, regulis Cardanicis cum fructu utimur," &c. Vol. III. p. 126. See also p. 54.

† "Elle etoit neanmoins d'autant plus essentielle, que l'expression de l'arc par le "sinus, fondée sur la serie connue, qui est l'intégrale de $\frac{dx}{\sqrt{1-x^2}}$, ne peut etre regardée "comme exacte, c'est a dire, comme representant à la fois tous les arcs qui ont le "même sinus; puisque cette serie ne represente evidemment qu'un seul des arcs qui "repondent au sinus dont il s'agit, savoir, le plus petit de ces arcs, celui qui est infé-"rieur, ou tout au plus egal, a 90 degrés. Cependant, c'est d'un autre côté une sorte "de paradoxe remarquable, que l'expression de l'arc par le sinus ne representant qu'un

mentions it as a remarkable paradox, that the series for the arc in terms of the sine represents only one arc, viz. the arc less than 90 degrees; whereas the series for the sine, produced by reversion from the former series, exhibits all possible arcs that have the same sine. I shall endeavour to solve this paradox, which, I think, originated partly from the introduction of geometrical considerations into an analytical investigation, by which the true derivation of certain expressions was concealed.

It has appeared that the equation $x=(2\sqrt{-1})^{-1}\{\varepsilon^{z\sqrt{-1}}-\varepsilon^{-z\sqrt{-1}}\}$ is true, when instead of z is put, $\theta+z$, or $2\theta+z$, ... or $n\theta+z$, or $\frac{\theta}{2}-z$, or $\frac{3\theta}{2}-z$ .... or $\frac{2n+1}{2}\theta-z$.

Now, if the fluxions of these equations are taken, and the equations cleared of exponential quantities, there results from each the same equation, to wit, $\dot{z}=\frac{\dot{x}}{\sqrt{1-x^2}}$. Hence, if the symbol $\int$ denotes the operation by which we are to ascend to the original equations from which $\dot{z}=\frac{\dot{x}}{\sqrt{1-x^2}}$ is derived, the only strict consequence from $\int\dot{z}=\int\frac{\dot{x}}{\sqrt{1-x^2}}$ is that $x=(\sqrt{-1})^{-1}\{\varepsilon^{z\sqrt{-1}}-\varepsilon^{-z\sqrt{-1}}\}$, or $=(2\sqrt{-1})^{-1}$

$$\{\varepsilon^{(\theta+z)\sqrt{-1}}-\varepsilon^{-(\theta+z)\sqrt{-1}}\},$$

or generally $=(2\sqrt{-1})^{-1}\{\varepsilon^{(n\theta+z)\sqrt{-1}}-\varepsilon^{-(n\theta+z)\sqrt{-1}}\}$, or

$$=(2\sqrt{-1})^{-1}\left\{\varepsilon^{\frac{2n+1}{2}\theta\sqrt{-1}}-\varepsilon^{\frac{-(2n+1)}{2}\theta\sqrt{-1}}\right\}.$$

" seul arc de 90 degrés au plus, l'expression du sinus par l'arc, qu'on peut deduire (par " la méthode du retour de suites) de l'expression de l'arc par le sinus, represente " exactement, etant poussée a l'infini, le sinus de tous les arcs possibles, plus petits " ou plus grands que 90°, et même que la circonference ou demi circonference, prise " tant de fois qu'on voudra. Je laisse à d'autres geometres, le soin d'éclaircir ce " mystère, ainsi que plusieurs autres," &c. p. 183.

Hence, to answer the equation $\dot{z} = \frac{\dot{x}}{\sqrt{1-x^2}}$,

$$x \text{ may} = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \&c.$$

$$\text{or } z' - \frac{z'^3}{1.2.3} + \frac{z'^5}{1.2.3.4.5} - \&c.$$

$$\text{or } z'' - \frac{z''^3}{1.2.3} + \frac{z''^5}{1.2.3.4.5} - \&c.$$

$\{z', z'', z''', \&c.$ representing $\theta + z$, $2\theta + z$, $3\theta + z$, &c.$\}$.
Suppose now it is necessary to deduce z, z', z'', &c. in terms of x and its powers, by reversion of series. What does the reversion of series mean? Merely this; a certain method or operation, according to which, one quantity being expressed in terms of another, the second may be expressed in terms of the first. Hence, in all similar series, the operation must be the same; consequently, the result, which is merely the exhibition of a formula, must be the same; so that, whatever is the series in terms of x, produced by *reversion* in

$x = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \&c.$ the same must be produced by reversion in $x = z' - \frac{z'^3}{1.2.3} + \frac{z'^5}{1.2.3.4.5} - \&c.$

$$\text{in } x = z'' - \frac{z''^3}{1.2.3} + \&c.$$

&c.

The series produced by reversion in these cases is, $x + \frac{x^3}{3.2} + \frac{3x^5}{5.8} + \&c.$ Hence it appears, that we know, a priori, that must happen which D'ALEMBERT considers as a paradox to have happened. Why this paradox found reception in the mind of this acute mathematician, I have stated, as my opinion, one cause to have been, an inattention, from geometrical considerations, to the real origin and derivation of certain expressions that appeared in the course of the calculation. Another cause I apprehend was, the want of precise notions on the force and

signification of the symbol $=$. It is true that its signification entirely depends on definition; but, if the definition given of it in elementary treatises be adhered to, I believe it will be impossible to shew the justness and legitimacy of most mathematical processes. It scarcely ever denotes numerical equality. In its general and extended meaning,* it denotes the result of certain operations. Thus, when from

$x = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5}$, $x = z' - \frac{z'^3}{1.2.3}$ &c.

z or z' is inferred $= x + \frac{x^3}{3.2} + \frac{3x^5}{5.8}$ &c. nothing is affirmed concerning a numerical equality; and all that is to be understood is, that $x + \frac{x^3}{3.2} + \frac{3x^5}{5.8} +$ &c. is the result of a certain operation performed on $x = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} -$ &c.

XIII. It appears then, that according to the reversion of series, z, z', z'', &c. must all be represented by the same series, proceeding according to the powers of x; but, if a form for z be required, which shall in all cases afford us a means of numerically computing its value, such a form must involve certain arbitrary quantities. These arbitrary quantities are to be determined by conditions which depend either on the original form of the equation between x and z, or on the nature of the object to which the calculus is applied.

Let now $\int \frac{\dot{x}}{\sqrt{1-x^2}}$ mean† $x + \frac{x^3}{3.2} + \frac{3x^5}{5.8} +$ &c.

* This is consistent with what I advanced in the Phil. Trans. for 1801. p. 99, concerning the meaning of the symbols $\times$ $+$, &c. It is beside my present purpose, to insist farther on the necessity of attaching precise notions to the symbols employed in calculation; and the subject deserves a separate and ample discussion.

† It is not so easy to prove as it may be imagined, that $\int \frac{\dot{x}}{\sqrt{1-x^2}} \cdot = x + \frac{x^3}{3.2} + \frac{3x^5}{5.8} +$ &c.

then, if z represent the arc of a circle, and x the sine, this equality* $z = x + \frac{x^3}{3.2} + \frac{3x^5}{5.8} +$ &c. is subject to restrictions, for x cannot exceed 1; consequently, the greatest value of z that can be determined from the equation, must be so determined by putting $x = 1$. Let $\pi = 1 + \frac{1}{3.2} + \frac{3}{5.8}$ &c.

Now, from the definition of sine and the nature of the circle, the arcs $2\pi - z$, $6\pi - z$ .... $(2n+1)\, 2\pi - z$ .... $4\pi + z$ .... $8\pi + z$ .... $4n\pi + z$, have the same sine; let these arcs be z, z', z'', z''', &c. and let $x + \frac{x^3}{3.2} + \frac{3x^5}{5.8} +$ &c. $= X$,

then $z' = 2\pi - X$, $z'' = 6\pi - X$, &c. or generally

$z''^{\cdots m} = \{2n+1\}\, 2\pi - X$, or $= 4n\pi + X$,

n any number of the progression 0, 1, 2, 3, 4, &c.

Or thus, from the conditions contained in the form of the equation between z and x,

since $\sqrt{1-x^2} = 2^{-1} . \{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\} = 1 - \frac{z^2}{1.2} +$ &c.

there is no possible value of z that answers the equation when x is > 1,

Let $\int \frac{\dot{x}}{\sqrt{1-x^2}} = X + \alpha$ $\therefore$ when z and x begin together,

$\alpha = 0$ and $z = X$.

But the equation $\frac{\dot{x}}{\sqrt{1-x^2}} = \dot{z}$ may be derived from $x = (2\sqrt{-1})^{-1}$

$\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\}$,

when instead of z is put $2\pi - z$, $6\pi - z$ .... $(2n+1)\, 2\pi - z$,

* In the expression $z = x + \frac{x^3}{3.2} + \frac{3x^5}{5.8} +$ &c. considered abstractedly from its origin and application, there is nothing that limits the value of x. Again, by applying the operation of reversion, x is represented by this form, $z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5}$ &c. But there is no method, I believe, of proving (I purposely exclude that unproved proposition that every equation has as many roots as dimensions) that instead of z in $x - z + \frac{z^3}{1.2.3} -$ &c. $= 0$, other quantities, as z', z'', &c. may be substituted.

$$\text{or } 4\pi+z \;\ldots\ldots\; 4n\pi+z.$$

Hence, z' or $2\pi-z=-X+\alpha$. Let $z=0 \therefore X=0 \therefore 2\pi=\alpha$.
Again, z'' or $6\pi-z=-X+\alpha$. Let $z=0 \therefore X=0 \therefore 6\pi=\alpha$.
Hence, the arbitrary quantity α may generally be represented by $(2n+1)2\pi$, or by $4n\pi \therefore z''^{\ldots m}=(2n+1)2\pi-X$,

$$\text{or } = 4n\pi+X.$$

XIV. I shall now shew, by a purely analytical process, what are the divisors of $x^n \mp a^n$. It seems a very strange and absurd method, to refer to the properties of geometrical figures, for the knowledge of the composition of analytical expressions.

Let $x=m^{\frac{1}{n}}\varepsilon^{\frac{z}{n}\sqrt{-1}} \therefore a^n=m\varepsilon^{z\sqrt{-1}} \therefore m=\frac{a^n}{\varepsilon^{z\sqrt{-1}}}$, and m will be always positive, if $\varepsilon^{z\sqrt{-1}}=1$. But (Art. XI.) the values of z that answer the equation $\varepsilon^{z\sqrt{-1}}=1$, are $o\theta, \pm\theta, \pm 2\theta, \pm 3\theta$, or generally $\pm s\theta$, s, any number of the progression 0, 1, 2, 3, &c.

Hence, $x=a\varepsilon^{\frac{\pm s\theta}{n}\sqrt{-1}}$ generally,

or values of x are a, $a\varepsilon^{\frac{\theta}{n}\sqrt{-1}}$, $a\varepsilon^{\frac{-\theta}{n}\sqrt{-1}}$, $a\varepsilon^{\frac{2\theta}{n}\sqrt{-1}}$, $a\varepsilon^{\frac{-2\theta}{n}\sqrt{-1}}$, &c.

$\therefore x^n-a^n=(x-a)\left(x^2-a\left\{\varepsilon^{\frac{\theta}{n}\sqrt{-1}}+\varepsilon^{\frac{-\theta}{n}\sqrt{-1}}\right\}+a^2\right)\left(x^2-a\left\{\varepsilon^{\frac{2\theta}{n}\sqrt{-1}}+\varepsilon^{\frac{-2\theta}{n}\sqrt{-1}}\right\}+a^2\right)$, &c. n being odd;

when n is even, (and of the form $2p$, p odd,) there must be a number (s) in the progression (0, 1, 2, 3, &c.) that $=\frac{n}{2}$; consequently, there must be a value of x, $a\varepsilon^{\frac{s\theta}{n}\sqrt{-1}}=a\varepsilon^{\frac{\theta}{2}\sqrt{-1}}$ $=-a$, since (Art. XI.) $\varepsilon^{2\pi\sqrt{-1}}$, or $\varepsilon^{\frac{\theta}{2}\sqrt{-1}}=-1$.

Hence, a quadratic divisor of x^n-a^n will be $(x-a).(x+a)$, or x^2-a^2; when n is even, and of the form $4p$, p even or odd,

there must be a number (s) in the progression (0, 1, 2, 3 ....) $=\frac{n}{4}$; consequently, there must be a value of x, $a\varepsilon^{\frac{\pm s\theta}{n}\sqrt{-1}} = a\varepsilon^{\frac{\pm\theta}{4}\sqrt{-1}} = a \times \pm\sqrt{-1}$, since (Art. XI.) $\varepsilon^{\pm\pi\sqrt{-1}}$, or $\varepsilon^{\frac{\pm\theta}{4}\sqrt{-1}} = \pm\sqrt{-1}$.

Hence, one quadratic divisor of $x^n - a^n$ will be of the form $x^2 + a^2 = (x + a\sqrt{-1}).\ (x - a\sqrt{-1})$; another, as it has been already shewn, will be of the form $x^2 - a^2$.

There are only n different divisors, for (n odd) the $(n-1)$th and nth divisors are comprised under the form $x = a\varepsilon^{\pm\frac{\overline{n-1}}{2n}\theta\sqrt{-1}}$; the succeeding divisors would be comprised under the form

$$x = a\varepsilon^{\pm\frac{\overline{n+1}}{2n}\theta\sqrt{-1}} = a\varepsilon^{\pm\theta\sqrt{-1}} \times \varepsilon^{\mp\frac{\overline{n-1}}{2n}\theta\sqrt{-1}}$$

$= a\varepsilon^{\mp\frac{\overline{n-1}}{2n}\theta\sqrt{-1}}$, (since $\varepsilon^{\pm\theta\sqrt{-1}} = 1$) the same as preceding form.

If $x^n + a^n = 0$, then $m = \frac{-a^n}{\varepsilon^{z\sqrt{-1}}}$, to have m always positive. Let $\varepsilon^{z\sqrt{-1}} = -1$, then (Art. XI.) the values of z are $\pm 2\pi, \pm 6\pi$...&c. Let $2\pi = \rho$, then generally $\varepsilon^{(2s+1)\rho\sqrt{-1}} = -1$; consequently, $x = a\varepsilon^{\frac{\pm(2s+1)}{n}\rho\sqrt{-1}}$, s any number of the progression 0, 1, 2, &c. or the values of x are $a\varepsilon^{\frac{\pm\rho}{n}\sqrt{-1}}$, $a\varepsilon^{\frac{\pm 3\rho}{n}\sqrt{-1}}$ &c.

or $x^n + a^n = \left(x^2 - a\left\{\varepsilon^{\frac{\rho}{n}\sqrt{-1}} + \varepsilon^{\frac{-\rho}{n}\sqrt{-1}}\right\} + a^2\right).\ \left(x^2 - a\left\{\varepsilon^{\frac{3\rho}{n}\sqrt{-1}} + \varepsilon^{\frac{-3\rho}{n}\sqrt{-1}}\right\} + a^2\right)$ &c.

When n is odd, there must be a number $(2s+1)$ in the progression (1, 3, 5, 7, &c.) $= n$; consequently, one value of x must

$= a\varepsilon^{p\sqrt{-1}} = -a$, or $x+a$ must be a divisor of $x^n + a^n$.

XV. Resolution of $x^{2n} - 2la^n x^n + a^{2n}$ into its quadratic factors $l \angle 1$.

Now, from the equation $x^n = a^n\{l \pm \sqrt{l^2-1}\} = \mathrm{A} \pm \mathrm{B}\sqrt{-1}$.

Let $x = m^{\frac{1}{n}} \varepsilon^{\frac{\pm z}{n}\sqrt{-1}}$ $\therefore m\varepsilon^{z\sqrt{-1}} = \mathrm{A}+\mathrm{B}\sqrt{-1}$, $m\varepsilon^{-z\sqrt{-1}} = \mathrm{A}-\mathrm{B}\sqrt{-1}$, $m = \sqrt{(\mathrm{A}^2+\mathrm{B}^2)}$, $2^{-1}\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\} = \frac{\mathrm{A}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}} = l$,

$$(2\sqrt{-1})^{-1}\{\varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}}\} = \frac{\mathrm{B}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}} = \sqrt{1-l^2};$$

but (Art. XI.) these equations are true, when instead of z are put $\theta+z$, $2\theta+z$, $3\theta+z$ .... generally $s\theta+z$.

Hence, the general value of x is $a\varepsilon^{\frac{\pm s\theta+z}{n}\sqrt{-1}}$, and the values of x are $a\varepsilon^{\frac{\pm z}{n}\sqrt{-1}}$, $a\varepsilon^{\frac{\pm\theta+z}{n}\sqrt{-1}}$, $a\varepsilon^{\frac{\pm 2\theta+z}{n}\sqrt{-1}}$,

or $x^{2n} - 2la^n x^n + a^{2n} = \left(x^2 - a\left\{\varepsilon^{\frac{z}{n}\sqrt{-1}} + \varepsilon^{\frac{-z}{n}\sqrt{-1}}\right\} + a^2\right) \times \left(x^2 - a\left\{\varepsilon^{\frac{\theta+z}{n}\sqrt{-1}} + \varepsilon^{\frac{-\theta+z}{n}\sqrt{-1}}\right\} + a^2\right) \times$ &c.

XVI. Such are the analytical processes according to which the resolutions of $x^n \mp a^n$, $x^{2n} \mp 2la^n x^n + a^{2n}$ are effected; and thence the fluents of $\frac{\dot{x}}{x^n \mp a^n}$, $\frac{\dot{x}}{x^{2n} - 2la^n x^n + a^{2n}}$, &c. &c. may be obtained, by resolving the fractions $\frac{1}{x^n - a^n}$ &c. into a series of partial fractions, of the form $\frac{\mathrm{A}x+\mathrm{B}}{x^2+2\alpha x+\alpha^2+\beta^2}$.

Since the above resolution of $x^n \mp a^n$ into its quadratic factors would, it appears to me, be strictly true, if such a curve as the circle had never been invented, nor its properties investigated, it is erroneous to suppose that the theorem of COTES is essentially necessary for the integration of certain differential forms.

That analytical science was advanced by the discovery of this theorem, is indeed true; but the circle and its lines were no farther useful or necessary, than as they afforded a mode of expressing, in geometrical language, an analytical truth. What is analytically expressed, may be analytically combined and resolved; and, if COTES, by the properties of figures, has expressed his discovery, it is because the mathematicians of the time in which he lived, were more skilful and dexterous with the geometrical method than with the analytical.

In order to demonstrate COTES's property of the circle, considered as such, one of two different methods must be pursued. Either let the demonstration be strictly geometrical, according to the method of the ancients, or as completely analytical as possible; that is, let the demonstration be effected by the analytical method, from as few fundamental principles as possible. I know not on what grounds of perspicuity and rigour, the propriety of a demonstration half geometrical, half algebraical, can be established; for, besides the want of symmetry in such a demonstration, in strictness of reasoning, a separate discussion is necessary, to shew the propriety and justness of the application of analysis to certain properties of extension demonstrated geometrically.

It is beside my present purpose, to inquire whether COTES's theorem can be demonstrated strictly after the method of the ancients: hitherto it has not been so demonstrated. To demonstrate it analytically, in the most simple and direct manner, we must proceed from as few fundamental principles as possible;* and give to the quantities concerned, their true and natural

* For the analytical demonstration, all that is necessary to be known, is what is proved in the 47th of the Elements.

representation. I think, therefore, the analytical demonstration in which the symbol $\sqrt{-1}$ is introduced, (for the cosine of an arc cannot be adequately and abridgedly represented in terms of the arc, except by means of the symbol $\sqrt{-1}$,) to be the most simple and direct that can be exhibited. I have endeavoured, in a former paper, to shew that demonstration with such symbols as $\sqrt{-1}$ may be strict and rigorous.

XVII. One or two more instances of the advantage accruing to calculation, from giving to quantities in analytical investigation their true analytical representation, I now offer, in the demonstrations of the series for the chord of the supplement of a multiple arc, in terms of the chord of the supplement of the simple arc, for the sine of the multiple arc, &c.

$$\text{Chord } 2\pi - z = (\sqrt{-1})^{-1}\left\{\varepsilon^{\frac{(2\pi - z)}{2}\sqrt{-1}} - \varepsilon^{\frac{-(2\pi - z)}{2}\sqrt{-1}}\right\},$$

$$= \varepsilon^{\frac{z}{2}\sqrt{-1}} + \varepsilon^{\frac{-z}{2}\sqrt{-1}}, \text{ since } \varepsilon^{\pi\sqrt{-1}} = \sqrt{-1}, \text{ and } \varepsilon^{-\pi\sqrt{-1}} = -\sqrt{-1}.$$

Again, chord $(2\pi - nz) = \varepsilon^{\frac{nz}{2}\sqrt{-1}} + \varepsilon^{\frac{-nz}{2}\sqrt{-1}}$. Let $\varepsilon^{\frac{z}{2}\sqrt{-1}} = \alpha$, $\varepsilon^{\frac{-z}{2}\sqrt{-1}} = \beta$ $\therefore$ $\alpha\beta = 1$; what we have to do then, is to find $\alpha^n + \beta^n$ in terms of $\alpha + \beta$; and, for facility of computation, a new mode of notation may be advantageously introduced, which requires a brief explanation only.*

* I had obtained the forms for chords nz, &c. given in the following pages, by actually expressing in terms of n and b, the coefficient of x^n, in the developement of the trinomial $\left\{1 + bx + x^2\right\}^m$, when the very admirable work of ARBOGAST, Du Calcul des Derivations, came to my hands. The great simplicity and convenience of his notation have caused me to adopt it, although it does not harmonize well with the fluxionary notation which I have employed in the present Paper.

By Art. V. $\varphi(x+o) = \varphi x + \mathrm{P}o + \frac{\mathrm{Q}o^2}{1.2} + \frac{\mathrm{R}o^3}{1.2.3}$ &c.
Let D be the note of the operation to be performed on φx, in order to deduce P, then $\mathrm{P} = \mathrm{D}\varphi x$, $\mathrm{Q} = \mathrm{DP} = \mathrm{DD}\varphi x = \mathrm{D}^2\varphi x$ &c.
Hence, $\varphi(x+o) = \varphi x + \mathrm{D}\varphi x o + \frac{\mathrm{D}^2\varphi x . o^2}{1.2} +$ &c.
or, representing $\frac{\mathrm{D}^n\varphi x}{1.2.3\ldots n}$ by $\underset{c}{\mathrm{D}}^n\varphi x$,
$\varphi(x+o) = \varphi x + \mathrm{D}\varphi x . o + \underset{c}{\mathrm{D}}^2\varphi x . o^2 + \underset{c}{\mathrm{D}}^3\varphi x . o^3 +$ &c.

To resume the demonstration:

$$\frac{1}{1+\alpha x} + \frac{1}{1+\beta x} = \frac{2+(\alpha+\beta)x}{1+(\alpha+\beta)x+\alpha\beta x^2} = \frac{2+bx}{1+bx+cx^2},$$

now $\frac{1}{1+\alpha x} = 1 - \alpha x + \alpha^2 x^2 \ldots\ldots \pm \alpha^n x^n \ldots$
and $\frac{1}{1+\beta x} = 1 - \beta x + \beta^2 x^2 \ldots\ldots \pm \beta^n x^n$
$\therefore$ term affected with x^n in developement of $\frac{1}{1+\alpha x} + \frac{1}{1+\beta x}$ is $\pm \{\alpha^n + \beta^n\}$.

Again, $\frac{2+bx}{1+bx+cx^2} = (2+bx)(1+bx+cx^2)^{-1}$
now $(1+bx)^{-1} = 1^{-1} + \mathrm{D}1^{-1}.bx + \underset{c}{\mathrm{D}}^2 1^{-1}.b^2x^2 + \underset{c}{\mathrm{D}}^3 1^{-1}.b^3x^3 +$ &c.
for b put $b+cx$, and for b^m, $\{b+cx\}^m$, or $b^m + \mathrm{D}b^m.cx + \underset{c}{\mathrm{D}}^2 b^m.c^2x^2$ &c.
then

$$(1+bx+cx^2)^{-1} = 1^{-1} + \mathrm{D}1^{-1}.bx + \mathrm{D}1^{-1}.bx.cx$$
$$+ \underset{c}{\mathrm{D}}^2 1^{-1}.b^2x^2 + \underset{c}{\mathrm{D}}^2 1^{-1}.\mathrm{D}b^2.cx^3 + \underset{c}{\mathrm{D}}^2 1^{-1}c^2x^4 \text{ \&c.}$$
$$+ \underset{c}{\mathrm{D}}^3 1^{-1}.b^3x^3 + \underset{c}{\mathrm{D}}^3 1^{-1}.\mathrm{D}b^3.cx^4 \text{ \&c.}$$
$$+ \underset{c}{\mathrm{D}}^4 1^{-1}.b^4x^4 \text{ \&c}$$

Hence, terms affected with x^n and x^{n-1} are

$$\underset{c}{\mathrm{D}}^n 1^{-1}.b^n + \underset{c}{\mathrm{D}}^{n-1} 1^{-1}.\mathrm{D}b^{n-1}c + \underset{c}{\mathrm{D}}^{n-2} 1^{-1}.\underset{c}{\mathrm{D}}^2 b^{n-2}c^2 +$$
$$\underset{c}{\mathrm{D}}^{n-3} 1^{-1}.\underset{c}{\mathrm{D}}^3 b^{n-3}c^3 + \text{\&c.}$$

and $\underset{c}{\mathrm{D}}^{n-1} 1^{-1}.b^{n-1} + \underset{c}{\mathrm{D}}^{n-2} 1^{-1}.\mathrm{D}b^{n-2}c +$ &c. Now, the mth term

from the beginning in first series, is $\underset{c}{D}^{n-m}\, 1^{-1} . \underset{c}{D}^{m}\, b^{n-m}\, c^{m}$;
which, n even and $m = \frac{n}{2} = \underset{c}{D}^{m}\, 1^{-1} . c^{m}$

$$n \text{ odd and } m = \frac{n-1}{2} = \underset{c}{D}^{m+1}\, 1^{-1} . \overline{m+1}\, b.$$

At these terms the series terminates; all the succeeding terms being equal 0, since $D^{m+1} b^{m}$, $D^{m+2} b^{m-1}$, are respectively $= \overline{m.m-1\, m-2} \ldots 3.2.1.0 = \overline{m-1\, m-2} \ldots 3.2.1.0$ and $= 0$.

Hence, the series written in a reverse order is (n even)

$$\underset{c}{D}^{m} 1^{-1}.c^{m} + \underset{c}{D}^{m+1} 1^{-1}.\underset{c}{D}^{m-1}.b^{m+1}.c^{m-1} + \underset{c}{D}^{m+2} 1^{-1}.\underset{c}{D}^{m-2} b^{m+2}.c^{m-2} \ldots \underset{c}{D}^{n} 1^{-1}.b^{n}$$

(n odd)

$$\text{or } \underset{c}{D}^{m+1} 1^{-1}.\underset{c}{D}^{m} b^{m+1}.c^{m} + \underset{c}{D}^{m+2} 1^{-1}.D^{m-1} b^{m+2}.c^{m-1} + \&c. \ldots\ldots\ldots \underset{c}{D}^{n} 1^{-1}.b^{n}$$

Now, $\underset{c}{D}^{n} 1^{-1} = \frac{-1.-2.-3\ldots-n}{1.2.3\ldots n} = 1$ (n even) or $= -1$ (n odd)
and $\therefore$ the former series becomes
$\pm b_{n} \pm D b^{n-1}.c \mp \underset{c}{D}^{2} b^{n-2}.c^{2} \pm$ &c. and consequently, the term affected with x^{n} in $(2 + bx)(1 + bx + cx^{2})^{-1}$

$$\text{is} \left\{ \begin{array}{l} \pm 2b^{n} \mp 2D\, b^{n-1}.c \pm 2\underset{c}{D}^{2} b^{n-2}.c^{2} \mp \&c. \\ \mp b^{n} \pm D\, b^{n-2}.bc \mp \underset{c}{D}^{2} b^{n-3}.bc^{2} \mp \&c. \end{array} \right\}$$

$$\text{or } \pm b^{n} \mp \frac{n}{n-1} D\, b^{n-1}.c^{2} \pm \frac{n}{n-2} D^{2} b^{n-2}.c^{2} \mp \frac{n}{n-3} D^{3} b^{n-3}.c^{3} \pm \&c.$$

$$\left\{ \begin{array}{l} \text{for, since } \underset{c}{D}^{m} b^{n-m-1} \times b = \frac{n-2m}{n-m} \underset{c}{D}^{m} b^{n-m} \\ \therefore \pm 2\underset{c}{D}^{m} b^{n-m} \mp \underset{c}{D}^{m} b^{n-m-1} \times b = \frac{n}{n-m} \underset{c}{D}^{m} b^{n-m} \end{array} \right\}$$

Hence, $\alpha^{n} + \beta^{n} = b^{n} - \frac{n}{n-1} D b^{n-1} + \frac{n}{n-2} \underset{c}{D}^{2} b^{n-2} -$ &c. c being $= \alpha\beta = 1$.

The law of the series is truly and unambiguously represented, by means of the symbol or note of derivation D; but, if it is required to express the law numerically, in terms of n, since

$$\underset{c}{D}^m b^{n-m} = \frac{(n-m)\ n-m-1)\ (n-m-2) \ldots\ldots (n-2m+1)}{1.2.3 \ldots\ldots\ldots\ldots m} b^{n-2m}$$

$$\alpha^n + \beta^n = b^n - n b^{n-2} + \frac{n.n-3}{1.2} b^{n-4} - \frac{n.(n-4)\ (n-5)}{1.2.3} b^{n-6} + \&c.$$

the series for the chord of the supplement of a multiple arc, in terms of the chord (b) of the supplement of the simple arc.

XVIII. Similar series may be found for the sines and cosines of multiple arcs; thus,

$$\cos. z = 2^{-1}\left\{\varepsilon^{z\sqrt{-1}} + \varepsilon^{-z\sqrt{-1}}\right\}, \cos. nz = 2^{-1}\left\{\varepsilon^{nz\sqrt{-1}} + \varepsilon^{-nz\sqrt{-1}}\right\}.$$

Now, $\alpha = \varepsilon^{z\sqrt{-1}}$ $\therefore \alpha^n = \varepsilon^{nz\sqrt{-1}}$. Let $\cos. z = p$,

$$\therefore \alpha + \beta = 2p = b,$$

$$\therefore \cos. nz = \frac{\alpha^n+\beta^n}{2} = \frac{1}{2}.\left(2^n p^n - n.2^{n-2}p^{n-2} + \frac{n.n-3}{1.2} 2^{n-4}p^{n-4} - \&c.\right)$$

$$= 2^{n-1}p^n - n.2^{n-3}p^{n-2} + \frac{n.n-3}{1.2} 2^{n-5}p^{n-4} - \&c. \quad \ldots\ldots$$

$$\text{or} = \tfrac{1}{2}\left\{(2p)^n - n.(2p)^{n-2} + \frac{n.n-3}{1.2}(2p)^{n-4} - \&c.\right\}$$

Suppose it were required to write the series in an inverse order: let n be even, then the series $b^n - \frac{n}{n-1} D b^{n-1}$ &c. terminates at a term $\frac{n}{n-m} \underset{c}{D}^m b^{n-m}$, $m = \frac{n}{2}$, and $\therefore \frac{n}{n-m} = 2$, and

$$\alpha^n + \beta^n = \pm 2 \mp \frac{n}{n-m+1} \underset{c}{D}^{m-1} b^{n-m+1} \pm \frac{n}{n-m+2} \underset{c}{D}^{m-2} b^{n-m+2} \mp \&c.$$

or, in terms of n,

$$= \pm 2 \mp \frac{n.n}{1.2.2} b^2 \pm \frac{\left(n.\frac{n}{2}.\frac{n}{2}+1.\frac{n}{2}-1\right)}{1.2.3.4} b^4 \mp \&c.$$

Consequently, $\cos. nz = \frac{\alpha^n+\beta^n}{2} = \pm 1 \mp \frac{n^2 p^2}{1.2} \pm \frac{n^2.(n^2-4)p^4}{1.2.3.4}$ &c.

Where the upper or lower sign takes place, as n is of the form $4s$, (s an even or odd number), or $2s$, (s an odd number); let n be odd, then the series terminates at a term $\frac{n}{n-m} \underset{c}{D}^m b^{n-m}$,

$m = \frac{n-1}{2}$, and $\therefore \frac{n}{n-m} . \underset{c}{D}^m b^{n-m} = nb$,

and $\alpha^n + \beta^n = \pm nb \mp \frac{n}{n-m+1} \underset{c}{D}^{m-1} b^{n-m+1} \pm$ &c.

or in terms of n

$$= \pm n b \mp \frac{n.(n^2-1)}{2^2.1.2.3} b^3 \pm \frac{n.(n^2-1)(n^2-9)}{2^4.1.2.3.4.5} b^5 \mp \&c.$$

Consequently,

$$\cos. nz = \frac{\alpha^n + \beta^n}{2} = \frac{1}{2}\left\{ \pm n b \mp \frac{n.(n^2-1)}{1.2.3} \frac{b^3}{2^2} \pm \frac{n.(n^2-1)(n^2-9)}{1.2.3.4.5} \frac{b^5}{2^4} \right\}$$

$$= \quad \pm np \mp \frac{n.(n^2-1)}{1.2.3} p^3 \pm \frac{n.(n^2-1)\,n^2-9}{1.2.3.4.5} p^5 - \&c.$$

Where the upper or lower sign is to be used, as n is of the form $(4s+1)$, or $4s+3$.

XIX. Again, sine $z = (2\sqrt{-1})^{-1} \left\{ \varepsilon^{z\sqrt{-1}} - \varepsilon^{-z\sqrt{-1}} \right\}$,

sine $nz = \left\{ 2\sqrt{-1} \right\}^{-1} \left\{ \varepsilon^{nz\sqrt{-1}} - \varepsilon^{-nz\sqrt{-1}} \right\}$,

$\therefore$ it is necessary to find $\alpha^n - \beta^n$ in terms of $\alpha - \beta$.

Let n be odd,

then term affected with x^n in developement of $\left\{ \frac{1}{1-\alpha x} + \frac{1}{1+\beta x} \right\}$

$= \alpha^n - \beta^n$, and $\frac{1}{1-\alpha x} + \frac{1}{1+\beta x} = \frac{2-(\alpha-\beta)x}{1-(\alpha-\beta)x - \alpha\beta x^2} = \frac{2-bx}{1-bx-cx^2}$,

and the term affected with x^n in the developement of $(2-bx)$

$$(1-bx-cx^2)^{-1} = b^n + \frac{n}{n-1} \mathrm{D} b^{n-1} c + \frac{n}{n-2} \overset{2}{\underset{c}{\mathrm{D}}} b^{n-2} c^2 - \&c.$$

or in terms of n $(c=1)$

$$= b^n + n b^{n-2} + \frac{n.(n-3)(n-4)}{1.2} b^{n-4} - \&c.$$

but sine $z = \frac{b}{2\sqrt{-1}} = p \therefore (b)^n = (2\sqrt{-1}p)^n = \pm 2^n\sqrt{-1}p^n$,

where the upper or lower sign is to be used, according as n is of the form $4s+1$, or $4s+3$. Hence,

$$\text{sine } nz = \frac{\alpha^n - \beta^n}{2\sqrt{-1}} = \pm 2^{n-1} p^n \mp 2^{n-3}.np^{n-2} \pm 2^{n-5}.\frac{n.(n-3)(n-4)}{1.2.3} p^{n-4}$$

&c.

If it is required to write the series in a reverse order, it is to be observed, that the series $b^n + \frac{n}{n-1} \mathrm{D} b^{n-1}$ &c. terminates at

a term $\frac{n}{n-m} \underset{c}{\text{D}}^m b^{n-m}$, $m = \frac{n-1}{2}$ $\therefore \frac{n}{n-m} \underset{c}{\text{D}}^m b^{n-m} = \frac{nb}{2}$;
consequently,

$$\alpha^n - \beta^n = \frac{nb}{2} - \frac{n}{n-m+1} \underset{c}{\text{D}}^{m-1} b^{n-m+1} \text{ \&c.}$$

or in terms of n,

$$= \frac{nb}{2} + \frac{n.(n+1)(n-1)}{1.2.3} \cdot \frac{b^3}{2^2} + \frac{n.(n+1)(n-1)(n+3)\,n-3}{1.2.3.4.5} \cdot \frac{b^5}{2^4} \text{ \&c.}$$

Hence, $\frac{\alpha^n - \beta^n}{2\sqrt{-1}}$ (sine nz) $= p - \frac{n.(n^2-1)}{1.2.3} - p^3 +$ &c.

XX. Let n be even, then term affected with x^n in developement of $\left\{\frac{1}{1-\alpha x} - \frac{1}{1+\beta x}\right\} = \alpha^n - \beta^n$.

Now $\frac{1}{1-\alpha x} - \frac{1}{1+\beta x} = \frac{(\alpha+\beta)x}{1-(\alpha-\beta)x - \alpha\beta x^2}$, and the term affected with x^{n-1} in the developement of $(1-bx-cx^2)^{-1}$ is $b^{n-1} + \text{D}b^{n-2}c + \underset{c}{\text{D}}^2 b^{n-2} c^2 + \underset{c}{\text{D}}^3 b^{n-3} c^3 +$ &c.

$\therefore$ term affected with x^n in $b' x\left\{1-bx-cx^2\right\}^{-1}$, $\left\{\alpha+\beta = b'\right\}$ is $b'\left\{b^{n-1} + \text{D}b^{n-2}c + \underset{c}{\text{D}}^2 b^{n-3} c^2 + \text{ \&c.}\right\}$

or in terms of n $(c=1)$

is $b'\left\{b^{n-1} + \left\{n-2\right\} b^{n-3} + \frac{(n-3)\,n-4}{1.2} b^{n-5} - \text{ \&c.}\right\}$

Hence, since sine $z = \frac{\alpha+\beta}{2\sqrt{-1}} = \frac{b}{2\sqrt{-1}} = p \therefore \frac{b^{n-1}}{\sqrt{-1}} = \pm 2^{n-1} p^{n-1}$,

and cosine $z = \frac{\alpha-\beta}{2} = \frac{b'}{2} = p' \therefore$ sine $nz = \frac{\alpha^n - \beta^n}{2\sqrt{-1}}$

$$= p'\left\{\pm 2^{n-1} p^{n-1} \mp 2^{n-3}.(n-2)p^{n-3} \pm \frac{2^{n-5}.(n-3)\,n-4}{1.2}\right\} \times$$

or $= p'\left\{\pm (2p)^{n-1} \mp (n-2)(2p)^{n-3} \pm \text{ \&c.}\right\}$

$\times p^{n-5} \mp$ &c. the upper signs taking place, if n is of the form $2s$ (s odd), the lower, if n is of the form $4s$, s even or odd.

If it is required to write the series in a reverse order, it is to be observed, that the series $b^{n-1} + \text{D}b^{n-2} +$ &c. terminates at a term $\underset{c}{\text{D}}^m b^{n-m-1}$, when $m = \frac{n}{2} - 1$; consequently, $\underset{c}{\text{D}}^m b^{n-m-1} = \frac{nb}{2}$,

and $\therefore \alpha^n - \beta^n = b'\left\{\frac{nb}{2} + \underset{c}{\text{D}}^{m-1} b^{n-m} + \text{ \&c.}\right\}$

$$= b'\left\{\frac{nb}{2} + \frac{\left(\frac{n}{2}+1\right)\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)}{1.2.3}\, b^3 + \&c.\right\}$$

$$= b'\left\{\frac{nb}{2} + \frac{n\,(n^2-4)}{1.2.4}\cdot\frac{b^3}{2^3} - \&c.\right\}$$

consequently, sine nz or $\frac{\alpha^n-\beta^n}{2\sqrt{-1}}$

$$= p'\left\{np - \frac{n\,.\,(n^2-4)}{1.2.3}\,p^3 + \frac{n\,.\,(n^2-4)\,(n^2-9)}{1.2.3.4.5}\,p^5 - \&c.\right\}$$

XXI. The sine nz (n even) may be expressed by series, in terms of the cosine of z;

thus, $\frac{1}{1-\alpha x} - \frac{1}{1-\beta x} = \frac{\{\alpha-\beta\}x}{1-\overline{\alpha+\beta}x+cx^2}$,

and, equating the terms affected with x^n in each developement, we shall have

$$\sin.\,nz = p\left\{(2p')^{n-1} - \frac{n-2}{1}\,(2p')^{n-3} + \frac{(n-3)\,(n-4)}{1.2}\,(2p')^{n-5} - \&c.\right\}$$

when n is even, a series may be found for sin. nz in terms of p (sin. z) only; but this series will not terminate as all the foregoing series do.

To find this series, expand $\sqrt{(1-p^2)} = p'$ into a series,

$$1^{\frac{1}{2}} - \mathrm{D}\,1^{\frac{1}{2}}p^2 + \underset{c}{\mathrm{D}^2}\;1^{\frac{1}{2}}p^4 - \&c.$$

then $\sin.\,nz = \left\{1 - \mathrm{D}\,1^{\frac{1}{2}}p^2 + \underset{c}{\mathrm{D}^2}\,1^{\frac{1}{2}}p^4 - \&c.\right\}\times\left\{np - \frac{n.(n^2-2^2)}{1.2.3.}\,p^3 + \&c.\right\}$

$= np + \mathrm{A}p^3 + \mathrm{A}_{,}p^5 + \mathrm{A}_{,,}p^7 + \&c.$

in which series, the law of the coefficients, or a general expression for $\mathrm{A}_{,,\ldots n}$ may be found. But it cannot now be done, without too long a digression from the present objects of inquiry.

From what has been done, the series* of the chord of the

* Demonstrations of these forms have been given by reversion of series, and by induction; which demonstrations are imperfect, since they do not exhibit the general law of the coefficients. See De Moivre Miscell. analytica. Epistola de Cotesii Inventis, &c. Newtoni Opera omnia. p. 306. Euler in Analyt. inf. Cap. 14. Waring has deduced the chord of the supplement of a multiple arc, in terms of the chord of the supplement of the simple arc, from his theorem for the powers of roots:

multiple arc may be found in terms of the chord of the simple arc; for, chord $nz = \left\{\sqrt{-1}\right\} \left\{ \varepsilon^{\frac{nz}{2}\sqrt{-1}} - \varepsilon^{-\frac{nz}{2}\sqrt{-1}} \right\}$.

XXII. In the above demonstrations, no formulas are borrowed from geometry; and the general law of the coefficients is clearly expressed; it is, I think, most conveniently expressed by means of the symbol or note of derivation D. The operation which this symbol indicates is as certain as any other operation, whether arithmetical or algebraical.

XXIII. The demonstrations and method of deduction given in this paper shew, I think, with sufficient evidence, the introduction of geometrical expressions and formulas into analytical investigation to be perfectly unnecessary. It has appeared likewise, that such introduction embarrasses investigation, and causes ambiguity, by concealing the true derivation of expressions; and, although I do not wish to give importance to my own observations, by supposing a greater confusion of notion to exist than really does, yet, I think, in what has been written and said, there may be detected a lurking opinion, that the value of certain expressions essentially demand the existence of geometrical curves and figures, and the investigation of their properties.

XXIV. In the Appendix to the *Arithmetica Universalis*, p. 200. 219. &c. NEWTON, with great clearness and force of argument, has shewn the distinction to be made between the order of classing curves, analytically considered, that is, defined

but the demonstration of the latter theorem is not, it appears to me, to be reckoned in the number of strict demonstrations. The only objection against the demonstration of the very learned and ingenious author of the Calcul des Derivations, is, that it is rather indirect, and blended with geometrical expressions and formulas.

by equations, and the order of classing them, considered as generated by description. Moreover, he animadverts on the custom of confounding the two sciences of algebra and geometry;* and, if any authority is attached to his assertion, that the two sciences ought not to be confounded together, the separation of geometry from algebra will thereby be equally urged as the separation of algebra from geometry. And it cannot be said with greater truth, that the simplicity of geometry is vitiated with algebraic equations, than that the simplicity of analysis is vitiated with geometrical forms and expressions. In fact, each science ought to be kept distinct; and be made to derive its riches from its proper sources.

XXV. It will not demand much meditation to be assured of this truth, that, in any mathematical investigation, the geometrical method, properly so called, is not essentially or absolutely necessary. The properties of extension and figure, to which this method has been especially appropriated, may be analytically treated; and here it is proper to state a distinction necessary to be made, between what may be called analytical geometry, and the application of analysis to geometry. The first does not suppose or require the existence of such a method as the geometrical; but, from a few fundamental principles, analytically investigates the properties of extension; whereas, in the latter, analysis is applied to propositions already established by the geometrical method: so that, strictly, to shew the justness and propriety of the application, a separate investigation is

* "Multiplicationes, divisiones, et ejusmodi computa, in geometriam recens introducta sunt: idque inconsulto, et contra primum institutum scientiæ hujus, &c. "—— Proinde hæ duæ scientiæ confundi non debent, &c.—— Et recentes, utramque confundendo, amiserunt simplicitatem in quâ geometriæ elegantia omnis consistit."

necessary. We find, however, in general, a vague analogy substituted, as a connecting principle between the two methods.

XXVI. The application of algebra to geometry, gives to DESCARTES the fairest title to fame for mathematical invention; yet the cause and nature of the benefit conferred on science by that application, seems to be indistinctly apprehended.* For, the Analytical Calculus, when applied to geometry, was not enriched with the truths of the latter science, because some connecting principle had been discovered, or some process invented by which the property of the two methods became common, and might, from one to the other, without formality be transferred; but because the investigation of certain properties could not proceed, without first improving the means by which they were to be investigated. These means DESCARTES improved: he found, when certain conditions in problems concerning extension were translated into the language of algebra, that the process of deduction with the general terms was slow and incommodious, because, such was the low state of the algebraic Calculus, the relation between the general terms had not been established. The aim and merit of DESCARTES's speculations is to have established this relation. If illustration were needed to make my meaning clear, I should say that DESCARTES, NEWTON, and D'ALEMBERT, benefited science precisely after the same manner. The first applied the analytical Calculus to extension; the second to motion; the third to the equilibrium, resistance, &c. of fluids. As the object of investigation became

* Thus far was the Analytical Calculus benefited by the existence of the geometrical method: certain properties of figure and extension, discovered by the latter, became to the former, objects of investigation.

more abstruse, it was found necessary to improve more and more the means or instrument of investigation.

XXVII. As the question concerning the respective advantages of the ancient geometry and modern analysis, is not foreign to the subject of this Paper, I shall briefly state it, and endeavour to afford the means of arriving therein at something like a precise determination.

The superiority of one method above another, must consist in being either more logically strict in its deductions, or more luminous, or more commodious for investigation. The discussion concerning the strictness and accuracy may, I conceive, be immediately put aside, since no method of deduction is essentially inaccurate; and, if in geometry the inferences are more strictly deduced than in the algebraic Calculus, the advantage is to be reckoned an accidental one, and arising from the great attention with which the former science has been cultivated.

One method may, however, be essentially more perspicuous and more commodious for investigation than another; or, in other words, the perspicuity and commodiousness of a method may depend on circumstances inherent in its nature and plan. Now, a person not sensible of the superior perspicuity of the geometrical method, would demand these circumstances, the necessary causes of perspicuity, to be pointed out to him; which might be done, by stating that geometry, instead of a generic term, employs, as a particular individual, the sign or representative of a genus; and that, as in algebra, the signs are altogether arbitrary, in geometry, they bear a resemblance to the things signified, and are called *natural* signs, since the figure of a triangle, or square, suggests to the mind the same tangible figure in Europe, that it does in America: and this resemblance

of the sign to the thing signified, is supposed to be the chief cause of the superior clearness of geometrical demonstration.* Another cause may perhaps be thought to exist in this circumstance, that whatever is demonstrated, of a triangle or other diagram, considered as the representative of all triangles and diagrams, is moreover demonstrated of that individual triangle or diagram. A third, and more satisfactory cause than the last, may be, that in investigation, for the purpose of preventing ambiguity and mistake, it is frequently necessary to recur from the sign to the thing signified; which is more easily done, the less general and arbitrary the modes of representation are; and, consequently, in geometry more easily than in algebra.

I do not pretend to have assigned, accurately, and all, the causes of perspicuity of geometrical reasoning. It may depend on certain intellectual acts and processes, which it is beyond the power of philosophy to explain. The circumstance, however, of the signs employed in geometry being *natural* signs, will prove its perspicuity only to a certain extent, and in certain cases. It must fail to prove it, when the properties of solids are treated geometrically; because the representation of solids on a plane by diagrams, is not a natural representation, that is, would not suggest to all minds the same tangible portion of extension.

It must fail likewise to prove it, in questions concerning radii of curvature, areas of curves, &c. or in all questions in which the fluxionary or differential Calculus is usually employed. The

* Does there not, however, here arise a consideration that takes away from the cause of the perspicuity of geometrical demonstration? For the reasoning with a diagram cannot be generally true, except the diagram be considered abstractedly, and independent of those peculiar and distinguishing properties that determine its individuality.

lines and mixtilinear triangles therein exhibited cannot be called natural signs, since they are only imperfect and inadequate representations of other imaginary lines and triangles, of which the mind must form what notion it can. Not, however, to infer want of perspicuity from inefficiency in the cause assigned, if we employ the geometrical method, or view its employment in investigation, concerning motion, curves, &c. it will not appear a perspicuous method; and, if instances of its obscurity were required of me, I could find them, even in the immortal work of the Principia. Whether we consider the fact, or speculate about the cause, I think the geometrical method can only be allowed to have superior evidence in investigations of a simple nature.

That the analytical calculus is more commodious for the deduction of truth than the geometrical, will not perhaps be contested; and, an examination into its nature, would shew why it is so well adapted for easy combination and extensive generalization. No language like the language of analysis, one of the greatest of modern mathematicians has observed, is capable of such elegance as flows from the developement of a long series of expressions connected one with the other, and all dependent on the same fundamental idea.

If we view what has been respectively done by each method, in the explanation of natural phenomena, the superiority of the one above the other will appear immense: yet the cultivators of geometry were men of consummate abilities, and possessed this great advantage, that the method or instrument of thought and reasoning which they employed had, during preceding times, received the greatest improvement. The analytical calculus, which has verified the principle of gravitation, was a hundred years ago in its infancy.

The question, then, concerning the respective advantages of the ancient geometry and modern analysis, may be comprised within a short compass. If mental discipline and recreation are sought for, they may be found in both methods; neither is essentially inaccurate; and, although in simple inquiries the geometrical has greater evidence, in abstruse and intricate investigation the analytical is most luminous: but, if the expeditious deduction of truth is the object, then I conceive the analytical calculus ought to be preferred. To arrive at a certain end, we should surely use the simplest means; and there is, I think, little to praise or emulate, in the labours of those who resolutely seek truth through the most difficult paths, who love what is arduous because it is arduous, and in subjects naturally difficult toil with instruments the most incommodious.

XXVIII. If in matters of abstract science deference is ever due to authority, it must be paid to that by which the study and use of the method of the ancients has been recommended. NEWTON has, however, brought forward no precise arguments in favour of synthesis; and it is easy to conceive, that he would be naturally attached to a method long known and familiar to him,* and by means of which he was enabled to connect his own theory of curvilinear motions, with the researches of the ancients on conic sections, and with HUYGENS's discoveries relative to central forces and the evolutes of curves.

The very ingenious and learned MATTHEW STEWART † endea-

* The circumstance of mathematicians having acquired a considerable dexterity in the management of the geometrical method, seems to be the reason why they endeavoured to explain the doctrine of logarithms (a subject purely algebraical) by the introduction of the properties of curves.

† Words are frequently stated in a delusive and imposing manner, not always

voured to shew, that the geometrical calculus was competent to the explanation of natural phenomena; and with astonishing perseverance applied it to many investigations in physical astronomy. The labours of such a man are not hastily to be judged: yet every one must determine for himself; and to me it seems, his reasonings, from their intricacy, call up so great a *contention of the mind*, that they prove, in no small degree, the unfitness of the geometrical method in all abstruse and intricate investigations.

XXIX. It may, however, be asked, are not there some subjects of inquiry to which the geometrical method is better adapted than the analytical? and is not the theory of angular functions one of these subjects?* I apprehend not: for, if the conditions

intentionally. Dr. STEWART, (Preface to Sun's Distance,) and after him his ingenious biographer, for the purpose of holding up the superior simplicity of the geometrical calculus, has said, that in order to understand his solution, a knowledge of the elements and conic sections only is requisite. But, in fact, the solution is effected by proposition heaped on proposition; and with equal truth and justness it might be said, that in order to understand the analytical solution, a knowledge only of common algebra is requisite; since the methods by which the solution is effected, are really and properly branches of algebra.

* D'ALEMBERT says, "there are cases in which analysis, instead of expediting, embarasses demonstration. These cases happen in the computation of angles: for angles can analytically be expressed only by their sines; and the expression of the sines of angles is often very complicated," &c. He adds, "that it must depend on mathematicians, whether the method of the ancients or the modern analysis is to be employed, since it would be difficult to give on this head exact and general rules." In the very case adduced, I think demonstration expedited by the analytical calculus; and, although $2^{-1}\left\{\varepsilon^{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}\right\}$ is not so speedily put down as cos. x; yet all processes of evolution, differentiation, integration, &c. are much more easily performed with the former expression than with the latter. Other instances of subjects of inquiry, to which the geometrical method is said to be peculiarly well adapted, have been adduced; but I still find no convincing reason, why a mathematician must submit to the necessity of

can be adequately and unambiguously stated in the general terms of algebra, then deduction with such terms may be strictly made, and expeditiously; since it is to be made according to a known and established process. I have shewn at some length, that reasoning may be conducted with terms which separately cannot be arithmetically computed: for the mere process of deduction, it is not necessary to have distinct and complete notions of the things signified by the general terms.

The principal object of the present paper is to shew, that the analytical calculus needs no aid from geometry, and ought to reject it, relying entirely on its own proper resources. By this means, it would gain perspicuity, precision, and conciseness; advantages not to be lightly estimated, by any one who has a regard to certainty and demonstration, or considers the bulk to which scientific treatises have of late years swelled.

In order to prove and illustrate the opinion I wished to establish, I directed my search to those cases which have been always thought to require the aid of the geometrical method. By a purely analytical process, I have traced the origin and derivation of certain fluxionary expressions, usually referred to logarithms and circular arcs. I have given demonstrations of the series for the sine of an arc in terms of the arc; of the analytical formula for the root of a cubic equation in the irreducible case; of the resolution of $x_n \mp a_n$ into quadratic factors; of the series for the chord, &c. of a multiple arc in terms of the simple arc, &c. which demonstrations, with as much confi-

learning half a series of truths by one method, and half by another. These considerations, however, depreciate the value of the geometrical method only in one point of view; for, after all, the finest exemplar of clear and accurate reasoning is contained in the works of EUCLID.

dence as I dare assume, knowing how fallaciously we judge of our own performances, I affirm to be strict and direct; established without artifices, and without foreign aid drawn from geometrical theorems and the properties of curves. In some parts of this paper, the subjects, for their importance, may be thought to be too slightly discussed; the fear of appearing prolix, has perhaps driven me into brevity and obscurity. In other parts, what I have advanced may be remote from common apprehension, or contrary to received opinion: but here I make no apology; for, what I have written, has been written only after long meditation, and from no love of singularity. "If I cannot add to truth," I do not desire distinction from "the "heresies of paradox."